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CRITICALLY PARTITIONABLE GRAPHS II

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A graph G is *l-degenerate* if $\delta(H) \leq l$ for all subgraphs H of G. A graph is said to be *(k, l)-unique* if $V(G)$ has exactly one partition into k subsets $V_1, \ldots, V_k$ such that each subset spans an l-degenerate graph. Extending a long sequence of results about graphs with high girth and high chromatic number, Bollobás and Thomason [4] proved the existence of (k, l)-unique graphs with arbitrarily large girth. This result is further strengthened here, where lower bounds are obtained on the vertex connectivity, edge connectivity and minimal degree of a (k, l)-unique graph and then examples of such graphs are displayed attaining these bounds and at the same time having large girth. Similar results are obtained for (k, l)-critical graphs and critically (k, l)-unique graphs.

The most obvious example which springs to mind of a graph with chromatic number k is a complete k-partite graph. Such graphs also have the property of being uniquely k-colourable, that is to say, there is only one colouring with k colours. These graphs have a number of what we might call 'rigidity' properties, such as low girth, high minimal degree and high connectivity. It is not at all straightforward to display k-chromatic graphs without these properties. As is well known, Erdös [6] established the existence of k-chromatic graphs of high girth by probabilistic means some twenty years ago, but it was only lately that Bollobás and Sauer [3] showed the existence of uniquely k-colourable graphs of large girth.

In recent years the concept of l-degenerate graphs has been introduced to extend the idea of colouring. A graph G is *l-degenerate* if $\max_{H \subset G} \delta(H) \leq l$. Note that a 0-degenerate graph is an independent set of vertices and a 1-degenerate graph is a forest. A *(k, l)-partition* $\{G_1, \ldots, G_k\}$ of G is a partition of G into k disjoint l-degenerate subgraphs $G_1, \ldots, G_k$ whose union spans G. The parameter $\chi_l(G)$ is the smallest k for which G has a (k, l)-partition. Thus $\chi_0(G) = \chi(G)$ and $\chi_1(G)$ is the point arboricity of G. Many colouring results extend to cover (k, l)-partitions, the most notable extension perhaps being Mitchem's extension of Brooks' theorem [9], stating that in all but a few special cases, which are described, the inequality $\chi_l(G) \leq [\Delta(G)/(l+1)]$ holds. (This result has been strengthened by Bollobás and Manvel [2].) For a survey of the theory of l-degenerate graphs the reader is referred to Simões-Pereira [11].

In this paper we consider three types of graphs. A *(k, l)-critical* graph $J = J_{k,l}$ is connected, has $\chi_l(J) = k$ and $\chi_l(J - e) = k - 1$ for each edge e of J. (A *vertex (k, l)-critical* graph J has $\chi_l(J) = k$ and $\chi_l(J - v) = k - 1$ for each vertex v of J.) A

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(k, l)-*unique* graph $L = L_{k,l}$ has a unique (k, l)-partition, and a *critically* (k, l)-*unique* graph $M = M_{k,l}$ is (k, l)-unique but $M - v$ is not (k, l)-unique for any vertex v of M. All other notation is that of Bollobás [1].

It was shown in [4] that there are graphs of high girth with a unique partition into k subgraphs with property P for various properties P, among them the property of being l-degenerate. Thus (k, l)-unique graphs of arbitrarily high girth exist, extending the work of [3] and [6] mentioned earlier. These (k, l)-unique graphs were still highly 'rigid' though in the sense of having large connectivity and minimal degree. Our object in this paper is to considerably strengthen the results of [4] by showing that as well as high girth we may also demand low minimal degree and low connectivity. More specifically, we find lower bounds for the connectivity κ, edge connectivity λ and minimal degree δ for each of the three types of graphs J, L and M mentioned above. We then construct graphs of these types which simultaneously have high girth and attain the given lower bounds for κ, λ and δ. (We describe our constructions only for $l \geq 1$. The construction of (k, l)-unique graphs readily extends to the case $l = 0$ but the construction of (k, l)-critical and critically (k, l)-unique graphs does not. This is because the constructions make use of large vertex $(2, l)$-critical graphs, which exist only if $l \geq 1$.)

We begin with the lower bounds on κ, λ and δ. Proofs are given of the bounds on λ. The proofs of the bounds on κ are omitted, being trivial and well known in the case $l = 0$. (The difference in the case $l = 1$ arise from the fact that a $(2, l)$-critical graph may have connectivity 1 if $l \neq 1$, but a $(2, 1)$-critical graph, *viz.* a cycle, has connectivity 2.) To prove the bounds on δ is equally easy, bearing in mind that if we take an l-degenerate graph and add a new vertex to it by at most l edges, the result is another l-degenerate graph.

Some cases of Theorems 1 to 3 can be found in Chartrand and Kronk [5], Kronk and Mitchem [7], Lick and White [8] and Simões-Pereira [10].

Theorem 1. *Let J be a (k, l)-critical graph. Then*

(i) $\kappa(J) \geq 1$, *and* $\kappa(J) \geq 2$ *if* $l = 1$;

(ii) $\lambda(J) \geq k - 1$ *and* $\lambda(J) \geq 2k - 2$ *if* $l = 1$; *and*

(iii) $\delta(J) \geq (k - 1)(l + 1)$.

Proof. Let $\lambda_l = 1$ for $l \neq 1$ and $\lambda_1 = 2$. We must show that $\lambda(J) \geq (k - 1)\lambda_l$. This is clearly true if $k = 2$.

Let S be a set of $\lambda(J)$ edges separating J into components G and H. Then G and H have $(k - 1, l)$-partitions $\{G_1, \ldots, G_{k-1}\}$ and $\{H_1, \ldots, H_{k-1}\}$ respectively. G_i is said to 'know' H_j if $V(G_i) \cup V(H_j)$ induces an l-degenerate subgraph in J. A 'marriage' is a permutation $j(1), j(2), \ldots, j(k - 1)$ of $1, 2, \ldots, k - 1$ such that G_i knows $H_{j(i)}$, $1 \leq i \leq k - 1$. Since a marriage defines a $(k - 1, l)$-partition of J no marriage can exist, so by the König–Hall theorem there are sets $\{G_1, \ldots, G_q\}$ and $\{H_q, \ldots, H_{k-1}\}$ such that G_i does not know H_j, $1 \leq i \leq q$, $q \leq j \leq k - 1$. But then

$G_i \cup H_j$ spans at least λ_l edges of S, so

$$\lambda(J) = |S| \geq q(k-q)\lambda_l \geq (k-1)\lambda_l.$$

To prove (iii), let $v \in V(J)$ and let $\{G_1, \ldots, G_{k-1}\}$ be a $(k-1, l)$-partition of $J-v$. Now $V(G_i) \cup \{v\}$ does not span an l-degenerate graph for any i (since $\chi_l(J) \geq k$) so v send at least $l+1$ edges to G_i. Hence v has degree at least $(k-1)(l+1)$. □

Theorem 2. *Let L be a (k, l)-unique graph. Then*

(i) $\kappa(L) \geq k-1$;

(ii) $\lambda(L) \geq \min\{(k-1)(l+1), \binom{k}{2}\}$, *and* $\lambda(L) \geq 2k-2$ *if* $l=1$; *and*

(iii) $\delta(L) \geq (k-1)(l+1)$.

Proof. Let S be a set of $\lambda(L)$ edges separating L into components A and B, and suppose that $|S| < (k-1)(l+1)$. Let $\{L_1, \ldots, L_k\}$ be the (k, l)-partition of L, with $A_i = A \cap L_i$, $B_i = B \cap L_i$, the L_i's being numbered so that $|A_i| = 0$, $i \leq a$, $|A_i| \neq 0$, $i > a$, $|B_j| \neq 0$, $j \leq k-b$, $|B_j| = 0$, $j > k-b$, for some integers a and b. Now for each part L_i and vertex $v \notin L_i$, there are at least $l+1$ edges between v and L_i, or else $\{v\} \cup L_i$ is l-degenerate. If $v \in A$ and $i \leq a$, these edges must all lie in S, and similarly if $v \in B$ and $i > k-b$. Hence $|S| \geq |A|\, a(l+1)$ and $|S| \geq |B|\, b(l+1)$. But $|A| \geq k-a$ and $|B| \geq k-b$. Since $|S| < (k-1)(l+1)$ we have $a=b=0$; that is, A_i and B_i are non-empty, $1 \leq i \leq k$. But now for each pair i and j, $1 \leq i, j \leq k$, there must be at least λ_l edges of S either between A_i and B_j or between A_j or B_i, for otherwise $\{L_1, \ldots, A_i \cup B_j, \ldots, A_j \cup B_i, \ldots, L_k\}$ is a new partition of L. Thus $\lambda(L) = |S| \geq \binom{k}{2}\lambda_l$, as required.

The bound on $\delta(L)$ is obtained by noting that if $\{L_1, \ldots, L_k\}$ is the (k, l)-partition of L, and $v \in V(L_i)$, then v sends at least $l+1$ edges to L_j $(j \neq i)$ because $L_j \cup \{v\}$ is not l-degenerate. □

Theorem 3. *Let M be a critically (k, l)-unique graph. Then*

(i) $\kappa(M) \geq k-1$;

(ii) $\lambda(M) \geq \min\{k(l+1), \binom{k}{2}\}$ *and* $\lambda(M) \geq \min\{2k, k(k-1)\}$ *if* $l=1$; *and*

(iii) $\delta(M) \geq k(l+1)$.

Proof. To prove (iii), let $\{M_1, \ldots, M_k\}$ be the (k, l)-partition of M, let $v \in V(M_i)$, and let $\{M'_1, \ldots, M'_k\}$ be a (k, l)-partition of $M-v$ distinct from $\{M_1, \ldots, M_i - v, \ldots, M_k\}$. Then v sends at least $l+1$ edges to each of $M'_1, \ldots, M'_k$; for otherwise $M'_j \cup \{v\}$ is l-degenerate for some j and $\{M'_1, \ldots, M'_j \cup \{v\}, \ldots, M'_k\}$ is a (k, l)-partition of M different from $\{M_1, \ldots, M_k\}$.

To get the lower bound on $\lambda(M)$, let A, B and S be as in the proof of Theorem 2, and suppose that $|S| < k(l-1)$. Once again $|S| \geq |A|\, a(l+1)$ and so $|A|\, a \leq k-1$; similarly $|B|\, b \leq k-1$. If $a = k-1$, then $|A| = 1$; but then $|S| \geq \delta(M) \geq$

$k(l+1)$. If $a=1$, then $|A|=k-1$; but then

$$|S|\geq|A|\,\delta(M)-\binom{|A|}{2}\geq k(l+1).$$

Consequently $a=b=0$, and so $|S|\geq\binom{k}{2}\lambda_l$ as before. □

We now proceed to our construction of graphs J, L and M of large girth attaining the lower bounds of Theorems 1 to 3. Although the full details of the constructions will be given, the proofs that the constructed graphs have the required properties will be condensed and in some cases omitted, since such proofs, though straightforward, would occupy a disproportionate amount of space. It is appropriate then to devote a line or two to describing the underlying ideas of the construction, to enable the reader who so wishes to complete the proofs without difficulty.

The first step is to construct a collection $\mathcal{H}$ of l-degenerate graphs of large girth, $l\geq 2$, with the property that any pair of them may be joined by a single edge to form a vertex $(2, l)$-critical graph. (If $l=1$ we take our graphs to be paths, and add two edges to form a cycle from two paths.) Now let A_i, B_j, $1\leq i, j\leq k$, be members of $\mathcal{H}$ (or the disjoint union of members of $\mathcal{H}$). A theorem from [4] tells us that there are graphs A and B of large girth whose only (k, l)-partitions are $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ respectively. Suppose now that G is a graph formed from A and B with a few edges added between A and B. If $\{G_1, \ldots, G_k\}$ is a (k, l)-partition of G, then this partition restricted to A must be $\{A_1, \ldots, A_k\}$, and restricted to B it must be $\{B_1, \ldots, B_k\}$. Hence the possibilities for $\{G_1, \ldots, G_k\}$ are limited by the ways we may choose A_i and B_j so that A_i, B_j and the extra edges between still form an l-degenerate graph. We may add edges for instance so that A_i and B_j span a vertex $(2, l)$-critical graph if $i\neq j$, so the only choice for $\{G_1, \ldots, G_k\}$ is $\{A_1\cup B_1, \ldots, A_k\cup B_k\}$. Of course, removing a vertex v from A_i or B_j renders $A_i\cup B_j$ l-degenerate, and now there are several choices of (k, l)-partitions of $G-v$. This means that G is critically (k, l)-unique. The construction of (k, l)-critical graphs follows a similar pattern.

The next lemma supplies us with the class $\mathcal{H}$ of graphs described in the previous paragraph. The second condition of the lemma concerns the number of edges $e(H)$ of a graph H and its maximal degree $\Delta(H)$. This condition will be needed for the application later of the theorem from [4].

Lemma. *Given $l\geq 2$ and $g\geq 3$ there are constants q and $n(l, g)$, and a class $\mathcal{H}$ of graphs with the following properties:*

(i) *if $n\geq n(l, g)$ there is a graph in $\mathcal{H}$ of order n;*

(ii) *if $H^n\in\mathcal{H}$, then $e(H)\geq l(n-q)$, $\Delta(H)\leq 7l$ and $g(H)\geq g$; and*

(iii) *if $H^n\in\mathcal{H}$ there is a vertex $x(H)\in V(H)$, such that for any two graphs H_1 and H_2 in $\mathcal{H}$, the graph $G=H_1\cup H_2+x(H_1)\,x(H_2)$ is vertex $(2, l)$-critical.*

Proof. By Theorem 1 of [4] there is an l-degenerate graph R of order r with $e(R) \geq l(r-p)$, $\Delta(R) \leq 6l$ and $g(R) \geq g$, where p is a constant. Because R is l-degenerate we may order the vertices of R as $x_1, \ldots, x_r$ such that if $T = T(t)$ is the subgraph of R induced by $\{x_1, \ldots, x_t\}$, $t \leq r$, then $\delta(T) = d_T(x_t) \leq l$. Since $e(R) \geq l(r-p)$ there is some constant q such that $\delta(T) = l$ for all except q values of t, and so $d_R(x) \geq l$ for all vertices x with at most q exceptions.

Choose R with $r \geq ql + q + 1$; thus there is an integer s satisfying

$$2r - q \geq s(l-1) \geq r + ql + 1.$$

Let S be a cycle of length s. Now $V(R)$ contains a set X, with $|X| = q$, such that $d_R(v) \geq l$ if $v \in V(R) - X$. Consider a function f on $V(R)$ satisfying $f(v) = \max\{1, l+1-d_R(v)\}$ for $v \in X$, $1 \leq f(v) \leq 2$ if $v \notin X$, and $f(y) = 2$ for some $y \notin X$. Since $q \leq \sum_{v \in X} f(v) \leq q(l+1)$ we can choose f so that $\sum_{v \in V(R)} f(v) = s(l-1)$, and so there is a bipartite multigraph B with vertex classes $V(S)$ and $V(R)$ such that $d_B(v) = l-1$ if $v \in V(S)$ and $d_B(v) = f(v)$ if $v \in V(R)$. Hence in the multigraph $F = S \cup B \cup R$ every vertex of R is joined to at least one vertex of S, y is joined to two vertices of S, $d_F(v) = l+1$ for each $v \in V(S)$ and $\Delta(F) \leq 6l + 2 \leq 7l$. If F has multiple edges we reckon $g(F) = 2$.

Let $\mathscr{H}^* = \mathscr{H}^*(r, s)$ be the set of all multigraphs formed from $S \cup R$ by adding a set E of $s(l-1)$ edges between S and R, and with the same degree sequence as F. We wish to show that $\max_{H \in \mathscr{H}^*} g(H) \geq g$. Suppose $\max_{H \in \mathscr{H}^*} g(H) = t < g$, and let $H^* \in \mathscr{H}^*$ have fewest distinct (not necessarily disjoint) t-cycles. Then there are vertices $a \in V(R)$, $b \in V(S)$, $ab \in E$ and a t-cycle containing ab. The number of vertices $w \in H^*$ whose distance $d(w, a)$ from a is at most t is at most

$$\Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{t-1} < \Delta^g,$$

where $\Delta = 7l$. Thus we may choose $c \in V(R)$ with $d(a, c) \geq t+1$ (assuming r is large enough), and so $d(b, c) \geq t$. Choose $d \in V(S)$ with $cd \in E$. Then $d(a, d) \geq t$, and so $d(b, d) \geq t-1$. Let

$$H^{**} = H^* - ab - cd + ad + bc.$$

Then $H^{**} \in \mathscr{H}^*$ has fewer t-cycles than H^* and $g(H^{**}) \geq t$. This contradicts our choice of H^*, so $t \geq g$.

Let $\mathscr{H}' = \bigcup_{r,s} \{H^* \in \mathscr{H}^*(r, s) \mid g(H^*) \geq g\}$. If $H' \in \mathscr{H}'$, choose a vertex $y \in V(R)$ of degree at least $l+2$ joined to a vertex $x \in V(S)$ and at least one other vertex of S. Let $H = H' - xy$, $\mathscr{H} = \{H = H' - xy \mid H' \in \mathscr{H}'\}$ and put $x(H) = x$.

By the construction of $\mathscr{H}$, (i) and (ii) are satisfied. It remains to verify (iii). Suppose that $G = H_1 \cup H_2 + x(H_1)x(H_2)$, where H_1 and H_2 are in $\mathscr{H}$. Let G^* be an induced subgraph of G with $\delta(G^*) \geq l+1$. Then $V(S_1) \cap V(G^*) \neq \emptyset$, since $\delta(R_1 \cap G^*) \leq l$. Therefore $V(S_1) \subseteq V(G^*)$, or else there is a vertex in $V(S_1) \cap V(G^*)$ of degree at most l in G^*. Since every vertex of R is joined to at least one vertex of S, it now follows that $V(R_1) \subseteq V(G^*)$, or else once again some vertex of $V(S)$ has degree at most l in G^*. Thus $V(H_1) \subseteq V(G^*)$, and likewise

$V(H_2) \subseteq V(G^*)$; hence $G^* = G$.

Thus G^* is vertex $(2, l)$-critical as required. $\square$

We are at last in a position to state and prove our main results, namely Theorems 4 to 6. We shall need the following result.

Proposition [4]. *Let $G_1, \ldots, G_k$ be l-degenerate graphs satisfying $n \leq |G_1| \leq \cdots \leq |G_k| \leq n+1$, $e(G_i) \geq l(n-c)$, $\Delta(G_i) \leq 7l$ and $g(G_i) \geq g$. Let U_i be a set of vertices of G_i with $|U_i| \leq c'$, where c' is some constant, such that if $u_i, \bar{u}_i \in U_i$, then $d_{G_i}(u_i, \bar{u}_i) \geq g-2$; let $U = \bigcup_{i=1}^{k} U_i$. Then if $n \geq n_0(k, l, g, c, c')$ there is a graph $G = G(G_i, k; U)$ with the following properties:*

(i) *G has a unique (k, l)-partition, namely $\{G_1, \ldots, G_k\}$; and*

(ii) *$g(G) \geq g$ and $d_G(u, \bar{u}) \geq g-2$ if $u, \bar{u} \in U$.*

This proposition is basically Theorem 2 of [4], where the property P of that theorem is the property of being l-degenerate. The only real difference is the requirement that $d_G(u, \bar{u}) \geq g-2$, but the proof of the theorem given in [4] can be easily modified to incorporate this, and we give no details here.

Theorem 4. *If $n \geq n(k, g)$ there is a $(k, 1)$-critical graph $J_{k,1}$ of order n with $\kappa(J) = 2$, $\lambda(J) = 2k-2$, $\delta(J) = 2k-2$ and $g(J) \geq g$. If $l \geq 2$ and $n \geq n(k, l, g)$ there is a (k, l)-critical graph $J_{k,l}$ of order n with $\kappa(J) = 1$, $\lambda(J) = k-1$, $\delta(J) = (k-1)(l+1)$ and $g(J) \geq g$.*

Proof. We describe the constructions of vertex (k, l)-critical graphs $J^*_{k,l}$ with the described properties, and it is then easy to see that $J^*_{k,l}$ must contain a subgraph $J_{k,l}$ with the desired properties. We also give a proof that $J^*_{k,l}$ is vertex (k, l)-critical for $l \geq 2$; this should give the flavour of the proof for $l = 1$ and also of the proofs of Theorems 5 and 6, which will be omitted.

Let $P_1, \ldots, P_{k-1}$ be paths. Let a_i and b_i be the end vertices of P_i and choose c_i and d_i in $V(P_i)$ with $d(x, y) \geq g-2$ for $x, y \in \{a_i, b_i, c_i, d_i\}$. Let $V = \bigcup_{i=1}^{k-1} \{a_i, b_i, c_i, d_i\}$. Let G be the graph $G(P_i, k-1; V)$ given by the Proposition and let $\tilde{G}$ be formed from $\tilde{P}_i$, $1 \leq i \leq k-1$ similarly. Form $J^*_{k,1}$ from $G \cup \tilde{G} \cup \{u, v\}$ by adding the edges ua_1, $u\tilde{a}_1$, ua_i, $u\tilde{a}_i$, ub_i, $u\tilde{b}_i$, $2 \leq i \leq k-1$ and vb_1, $v\tilde{b}_1$, vc_i and vd_i, $2 \leq i \leq k-1$.

Now suppose $l \geq 2$. Take graphs $H_1, \ldots, H_{k-1} \in \mathscr{H}$ (see Lemma) and form $G = G(H_i, k-1; V)$ where $V = \{x(H_i) \mid 1 \leq i \leq k-1\}$. Choose graphs $\tilde{H}_1, \ldots, \tilde{H}_{k-1} \in \mathscr{H}$. Let $\tilde{Y}_i$ be the set of vertices joined to $x(\tilde{H}_i)$ in $\tilde{H}_i$ and let $\tilde{V} = \bigcup_{i=1}^{k-1} \tilde{Y}_i$. Let $\tilde{G} = G(\tilde{H}_i - x(\tilde{H}_i), k-1; \tilde{V})$.

Form $J^*_{k,l}$ from $G \cup \tilde{G} \cup \{w\}$ by adding the edges $wx(H_i)$, $1 \leq i \leq k-1$, and $w\tilde{v}$ for each $\tilde{v} \in \tilde{V}$. (See Fig. 1.)

Suppose that $J^* = J^*_{k,l}$ has a $(k-1, l)$-partition $\{J^*_1, \ldots, J^*_{k-1}\}$. Then this partition induces the partitions $\{H_1, \ldots, H_{k-1}\}$ and $\{\tilde{H}_1 - x(\tilde{H}_1), \ldots, \tilde{H}_{k-1} - x(\tilde{H}_{k-1})\}$

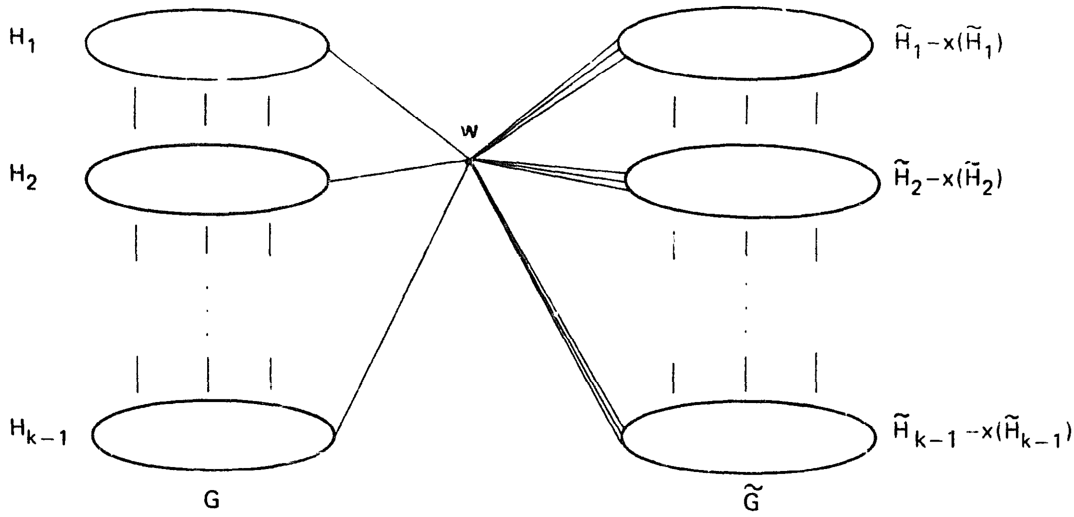

Fig. 1. A (k, l)-critical graph.

of G and $\tilde{G}$ respectively. We may suppose that $w \in J_1^*$, $H_1 = G \cap J_1^*$ and $\tilde{H}_1 - x(\tilde{H}_1) = \tilde{G} \cap J_1^*$. Then J_1^* is the subgraph of J^* spanned by H_1, $\tilde{H}_1 - x(\tilde{H}_1)$ and w. But $\tilde{H}_1 - x(\tilde{H}_1) \cup \{w\}$ spans a graph isomorphic to $\tilde{H}_1$, and so J_1^* is isomorphic to $H_1 \cup \tilde{H}_1 + x(H_1)x(\tilde{H}_1)$, a contradiction. Thus $\chi_l(J^*) \geq k$.

Now let $v \in V(J^*)$. Then either

(i) $v \in V(H_i)$, when

$$\{H_1 \cup \tilde{H}_1 - x(\tilde{H}_1), \ldots, H_i - v + w \cup \tilde{H}_i - x(\tilde{H}_i), \ldots, H_{k-1} \cup \tilde{H}_{k-1} - x(\tilde{H}_{k-1})\}$$

is a $(k-1, l)$-partition of $J^* - v$,

(ii) $v \in V(\tilde{H}_i)$, when

$$\{H_1 \cup \tilde{H}_1 - x(\tilde{H}_1), \ldots, H_i + w \cup \tilde{H}_i - x(\tilde{H}_i) - v, \ldots, H_{k-1} \cup \tilde{H}_{k-1} - x(\tilde{H}_{k-1})\}$$

is a $(k-1, l)$-partition of $J^* - v$, or

(iii) $v = w$, when

$$\{H_1 \cup \tilde{H}_1 - x(\tilde{H}_1), \ldots, H_{k-1} \cup \tilde{H}_{k-1} - x(\tilde{H}_{k-1})\}$$

is a $(k-1, l)$-partition of $J^* - v$.

Thus for any vertex v of J^*, $\chi_l(J^* - v) \leq k - 1$ so J^* is vertex (k, l)-critical. □

Theorem 5. *If $n \geq n(k, g)$ there is a $(k, 1)$-unique graph $L_{k,1}$ of order n with $\kappa(L) = k-1$, $\lambda(L) = 2k-2$, $\delta(L) = 2k-2$ and $g(L) \geq g$. If $l \geq 2$ and $n \geq n(k, l, g)$ there is a (k, l)-unique graph $L_{k,l}$ of order n with $\kappa(L) = k-1$, $\lambda(L) = \min\{(k-1)(l+1), \binom{k}{2}\}$, $\delta(L) = (k-1)(l+1)$ and $g(L) \geq g$.*

Proof. Let P_i be a path with endvertices x_i and y_i, $1 \leq i \leq k$. Let $V = \bigcup_{i=1}^k \{x_i, y_i\}$. Form $G = G(P_i, k; V)$. Take two copies of G, namely G and G^*, and form $L_{k,1}$ by identifying y_i with x_i^*, $1 \leq i \leq k-1$, and by adding a vertex v with edges vx_i and vy_i, $1 \leq i \leq k-1$.

For $l \geq 2$, select graphs $H_1, \ldots, H_k \in \mathscr{H}$ and vertex $(2, l)$-critical graphs $G_1, \ldots, G_{k-1}$ (also given by the Lemma). Select vertices $y_i \in G_i$ of degree $l+1$ and let Y_i be the set of vertices joined in G_i to y_i. Let $Y = \bigcup_1^{k-1} Y_i$, and let $V = Y \cup \{x(H_i) \mid 1 \leq i \leq k\}$. Let $H_i^* = H_i \cup (G_i - y_i)$, $1 \leq i \leq k-1$, and $H_k^* = H_k$. Form $G = G(H_i^*, k; V)$. Let $\tilde{G} = G(G_i, k; V)$, where $\tilde{G}_1$ is a member of $\mathscr{H}$ and $\tilde{G}_i$ is the union of $\tilde{H}_i^1, \ldots, \tilde{H}_i^{i-1}$, with $H_i^j \in \mathscr{H}$. Then $L_{k,l}$ is constructed from $G \cup \tilde{G} \cup \{v\}$ by joining $x(H_i)$ to $x(\tilde{H}_j^i)$ for $j \geq i+1$, and by joining v to every vertex in the set Y. (See Fig. 2.) $\square$

Theorem 6. *For $n \geq n(k, g)$ there is a critically $(k, 1)$-unique graph $M_{k,1}$ of order n with $\kappa(M) = k-1$, $\lambda(M) = \min\{2k, k(k-1)\}$, $\delta(M) = 2k$ and $g(M) \geq g$. For $l \geq 2$ and $n \geq n(k, l, g)$ there is a critically (k, l)-unique graph $M_{k,l}$ of order n with $\kappa(M) = k-1$, $\lambda(M) = \min\{k(l+1), \binom{k}{2}\}$, $\delta(M) = k(l+1)$ and $g(M) \geq g$.*

Proof. Let $P_1, \ldots, P_{k-1}$ be paths with endvertices x_i and y_i, $1 \leq i \leq k-1$, and let the midpoint of P_1 be $\bar{z}$; let Z_1 be the set of vertices adjacent to $\bar{z}$ in P_1. Let $S_2, \ldots, S_k$ be paths whose set of endvertices is Z^*; let $Z = Z_1 \cup Z^*$. Let $G_1 = P_1 - \bar{z}$, and $G_i = P_i \cup S_i$, $2 \leq i \leq k$. Let the midpoint of S_i be w_i, and define $V = Z \cup \{x_i, y_i, w_i \mid 1 \leq i \leq k\}$. Then form $G = G(G_i, k; V)$. Let $Q_1, \ldots, Q_{k-1}$ be paths with endvertices a_i and b_i. Let R_i^j be a path with endvertices c_i^j and d_i^j, $1 \leq j \leq i-1$, $2 \leq i \leq k$. Let the midpoint of R_k^{k-1} be $\bar{u}$, and let U be the set of vertices adjacent to $\bar{u}$ in R_k^{k-1}. We define $H_1 = Q_1$ and for $2 \leq i \leq k-1$,

$$H_i = Q_i \cup \bigcup_{j=1}^{i-1} R_i^j, \qquad H_k = \bigcup_{j=1}^{k-1} R_k^j - \{\bar{u}\}.$$

Let $W = U \cup \{a_i, b_i, c_i^j, d_i^j \mid 1 \leq j \leq i-1, 1 \leq i \leq k\}$, and form $H = G(H_i, k; W)$. Define M by adding to $G \cup H \cup \{v, w\}$ the following edges: $x_j c_i^j$, $y_j d_i^j$, $j < i \leq k$, $1 \leq j \leq k-1$, va_i, vb_i, $1 \leq i \leq k-1$, vu for each $u \in U$ and wz for each $z \in Z$. Then M contains a critically $(k, 1)$-unique graph M^*. To form $M_{k,1}$ take two disjoint copies M^* and M^{**} of M^*, and identify w_i^* with w_i^{**}, $2 \leq i \leq k$.

To construct $M_{k,l}$, $l \geq 2$, let $A_1, \ldots, A_{k-1}$ be graphs in $\mathscr{H}$. Let B_i, $2 \leq i \leq k$, be a vertex $(2, l)$-critical graph given by the Lemma, and let u_i be a vertex of B_i of degree $l+1$. Let the vertices in B_i adjacent to u_i be U_i. Choose $u_1 \in V(A_1)$ of degree $l+1$, joined to the set $U_1 \subseteq V(A_1)$. Let $U = \bigcup_{i=1}^k U_i$. Let $G_1 = A_1 - u_1$ and let $G_i = A_i \cup B_i - u_i$, $2 \leq i \leq k$. Let $V = U \cup \{x(A_i) \mid 1 \leq i \leq k-1\}$ and form $G = G(G_i, k; V)$. Let C_i^j be a member of $\mathscr{H}$, $1 \leq j \leq i-1$, $2 \leq i \leq k$. Let $D_1, \ldots, D_{k-1}$ be $(2, l)$-critical graphs given by the Lemma and choose $v_i \in V(D_i)$ of degree $l+1$. Let W_i be the set of vertices in D_i joined to v_i. Select $y \in V(C_k^{k-1})$ of degree $l+1$, and let W_k be the set of vertices in C_k^{k-1} joined to y. Define

$$W = \bigcup_{i=1}^{k} W_i \cup \{x(C_i^j) \mid 1 \leq j \leq i-1, 2 \leq i \leq k\}.$$

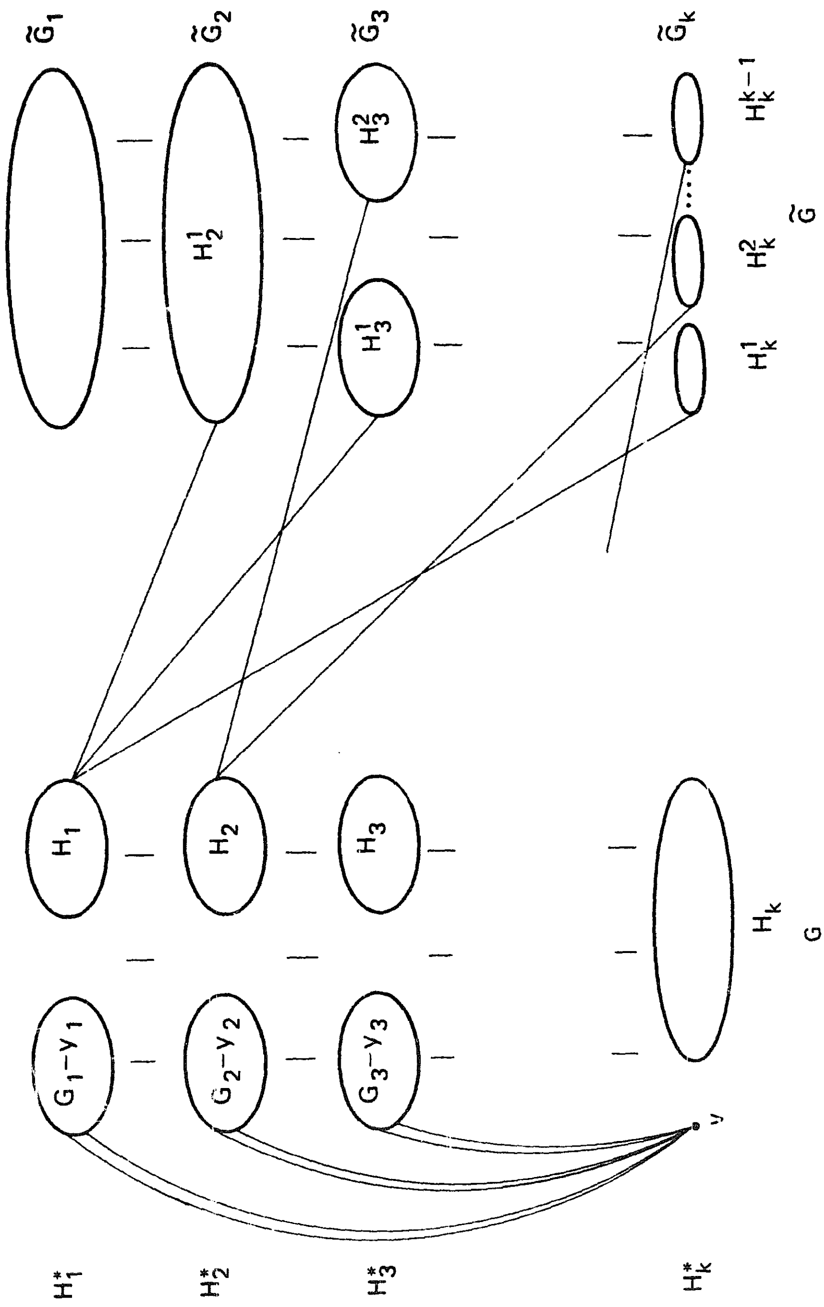

Fig. 2. A (k, l)-unique graph.

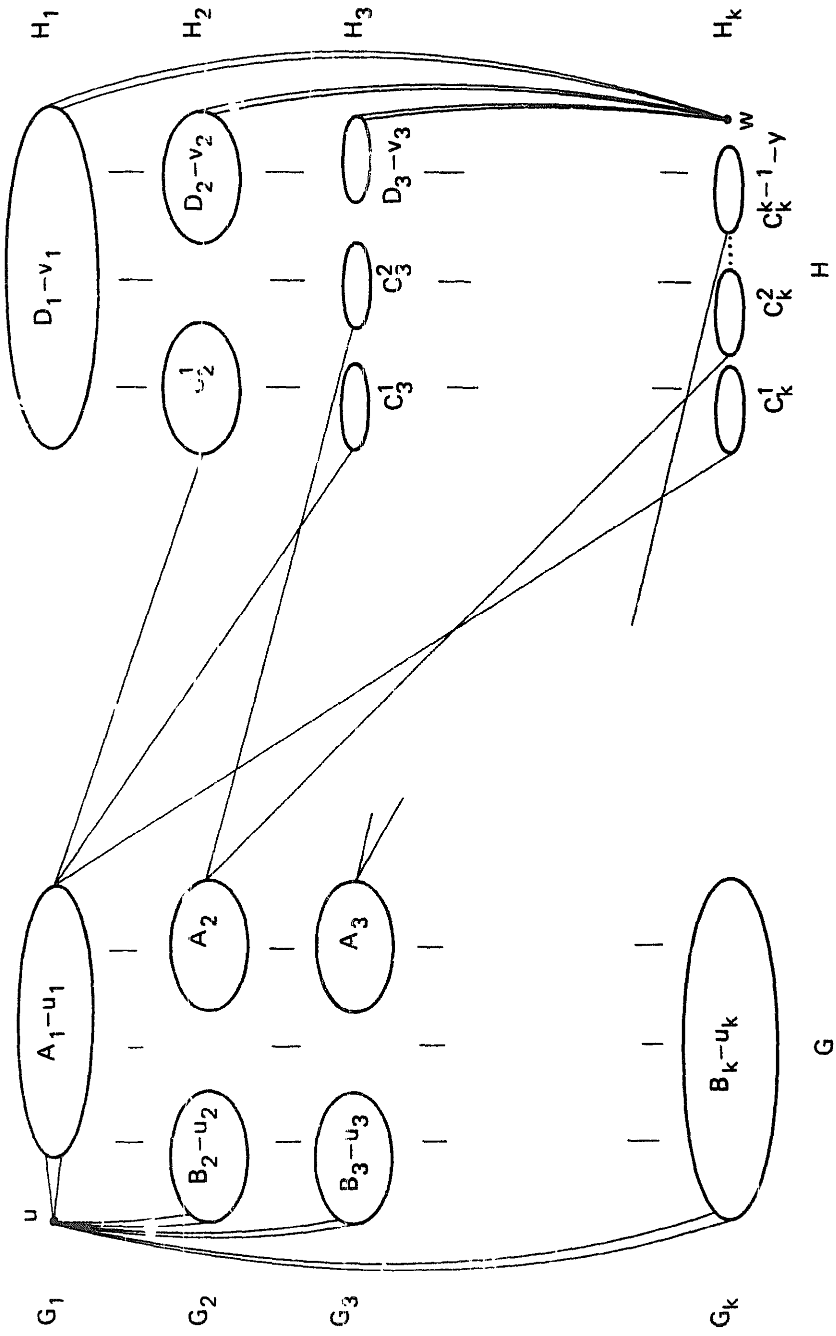

Fig. 3. A critically (k, l)-unique graph.

Now form $H_1 = D_1 - v_1$, $H_i = [D_i - v_i] \cup \bigcup_{j=1}^{i-1} C_i^j$ for $2 \leq i \leq k-1$, and $H_k = \bigcup_{j=1}^{k-1} C_k^j - y$.

Define $H = G(H_i, k; W)$. Then $M_{k,l}$ is constructed from $G \cup H \cup \{u, w\}$ by adding the edges $x(A_j)x(C_i^j)$, $j+1 \leq i \leq k$, $1 \leq j \leq k-1$, joining u to each vertex in U and joining w to each vertex in W_i, $1 \leq i \leq k$. (See Fig. 3.) □

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